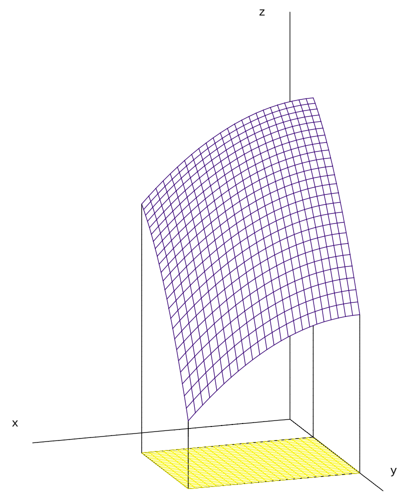
C3M10

Surface Area

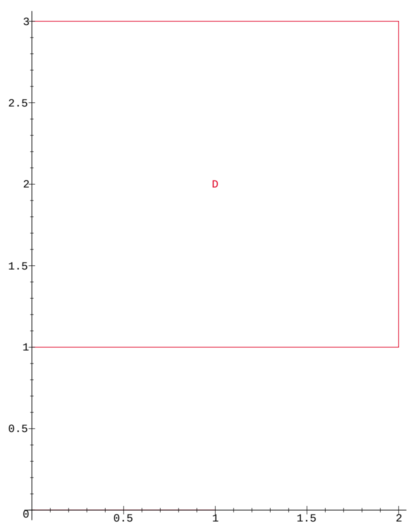
There are two ways to define a surface in 3-space. If we think of $z = f(x, y)$ with a domain D in the plane (2-space, or $\mathbb{R}^2$), then we visualize the surface as those points with coordinates $(x, y, f(x, y))$ which "lie over" the set D which is in the xy-plane, shown here in yellow. This is the first approach.



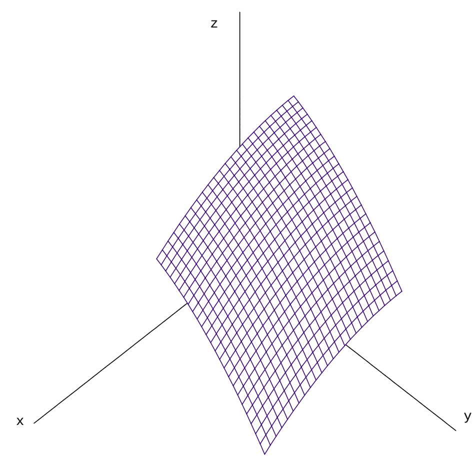
This is actually a special case of defining a surface parametrically. If D is a set in $\mathbb{R}^2$ and g is defined as

$$g\colon D \to \mathbb{R}^3 \qquad g(u, v) = \begin{pmatrix} g_1(u, v) \\ g_2(u, v) \\ g_3(u, v) \end{pmatrix}$$

then our surface is defined as the image $g(D) = S$.

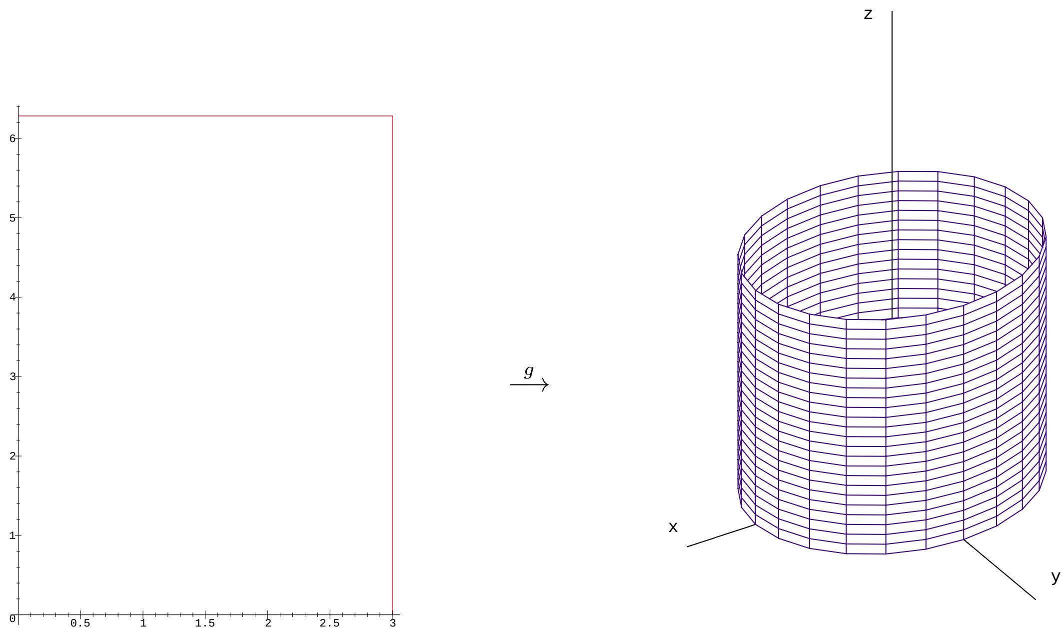


$\xrightarrow{g}$



Example 1. Suppose $z = f(x, y)$ with domain D. Define $g(u, v) = \begin{pmatrix} u \\ v \\ f(u, v) \end{pmatrix}$ where it is obvious that u and v play the role of x and y.

Example 2. Suppose that $D = \{(u, v) :\ 0 \le u \le 3, 0 \le v \le 2\pi\}$ and $g(u, v) = \begin{pmatrix} 2\cos(v) \\ 2\sin(v) \\ u \end{pmatrix}$. Maybe you would be more comfortable with $g(z, \theta) = \begin{pmatrix} 2\cos(\theta) \\ 2\sin(\theta) \\ z \end{pmatrix}$. As θ changes from 0 to 2π, The x and y coordinates go around a circle of radius 2. The variable z is the vertical coordinate and assumes all values between 0 and 3. We have parameterized the curved surface of a cylinder of radius 2 and height 3.



Example 3. Consider

$$g(u,v) = \begin{pmatrix} u\cos v \\ u\sin v \\ u \end{pmatrix} \qquad \begin{matrix} 0 \le u \le 2 \\ 0 \le v \le 2\pi \end{matrix}$$

or equivalently,

$$g(r,\theta) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ r \end{pmatrix} \qquad \begin{matrix} 0 \le r \le 2 \\ 0 \le \theta \le 2\pi \end{matrix}$$

If r is held constant and θ varies, then you obtain a circle of radius r in the plane $z = r$. If θ is held constant at $\theta = 0$, and r varies, the result is a line in the vertical plane $y = 0$ and the slanted plane $z = x$. Now rotate that around the z-axis and you will understand how we have parameterized an inverted cone with base radius 2 and height 2. ($x^2 + y^2 = z^2, 0 \le z \le 2$). Look at the x and y components of g and you will see that a disk of radius 2 is parameterized. But as r increases, so does z, which yields a cone instead of a disk.

If we compute $\frac{\partial g}{\partial u}$ and $\frac{\partial g}{\partial v}$ then we obtain the vector valued functions

$$\frac{\partial g}{\partial u} = \begin{pmatrix} \frac{\partial g_1}{\partial u} \\ \frac{\partial g_2}{\partial u} \\ \frac{\partial g_3}{\partial u} \end{pmatrix} \qquad \text{and} \qquad \frac{\partial g}{\partial v} = \begin{pmatrix} \frac{\partial g_1}{\partial v} \\ \frac{\partial g_2}{\partial v} \\ \frac{\partial g_3}{\partial v} \end{pmatrix}$$

which may be regarded as tangent vectors at a point on the surface. Visualize the parallelogram that these two tangent vectors would generate. How would we find the area of that parallelogram? Simple! Take their cross product and find its length! We have a name for $\dfrac{\partial g}{\partial u} \times \dfrac{\partial g}{\partial v}$, it is the **fundamental cross product**. The parallelogram we generated may be seen loosely as the image of a unit square in the domain, D, generated by a translation of the unit vectors, $\vec{i}$ and $\vec{j}$. The length of the fundamental cross product is like a magnification factor for the square whose area is 1 unit. Multiplying by $du\,dv$ we obtain $d\sigma = \left\| \frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} \right\| du\,dv$, which is regarded as an element of surface area. If we integrate $d\sigma$ over the domain, D, then we obtain the area of our surface S, $\sigma(S)$.

$$\sigma(S) = \iint_D d\sigma = \iint_D \left\| \frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} \right\| du\,dv$$

Example 1, revisited. Let's compute the fundamental cross product for g.

$$g(u,v) = \begin{pmatrix} u \\ v \\ f(u,v) \end{pmatrix} \qquad \frac{\partial g}{\partial u} = \begin{pmatrix} 1 \\ 0 \\ f_u \end{pmatrix} \qquad \text{and} \qquad \frac{\partial g}{\partial v} = \begin{pmatrix} 0 \\ 1 \\ f_v \end{pmatrix}$$

The cross product and $d\sigma$

$$\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} = \begin{vmatrix} 1 & 0 & f_u \\ 0 & 1 & f_v \\ \vec{i} & \vec{j} & \vec{k} \end{vmatrix} = \langle -f_u, -f_v, 1 \rangle \qquad d\sigma = \left\| \frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} \right\| du\, dv = \sqrt{(f_u)^2 + (f_v)^2 + 1}\, du\, dv$$

which explains the formula for surface area that you see when given $z = f(x,y)$ over D.

$$\sigma(S) = \iint_D \sqrt{(f_x)^2 + (f_y)^2 + 1}\, dx\, dy$$

Example 4. Find the surface area of that portion of the plane $2x+3y+4z = 36$ that lies over the rectangle $D = \{(x,y) : \ 0 \le x \le 3, 0 \le y \le 2\}$.

Solving for z, $z = 9 - x/2 - 3y/4$. So $\sqrt{(f_x)^2 + (f_y)^2 + 1} = \sqrt{(-1/2)^2 + (-3/4)^2 + 1} = \sqrt{29/16} = \sqrt{29}/4$. So,

$$\sigma(S) = \int_0^3 \int_0^2 \frac{\sqrt{29}}{4}\, dy\, dx = \frac{\sqrt{29}}{4} \cdot 2 \cdot 3 = \frac{3\sqrt{29}}{2}$$

A little thought here allows us to realize that because the length of the fundamental cross product is a constant, if we know the area $\sigma(D)$ of any domain D, the area of the portion of the given plane 'above' D is always just $\frac{\sqrt{29}}{4} \cdot \sigma(D)$. This explains why we referred earlier to the length of the fundamental cross product as a magnification factor.

Example 5 (Maple): Use Maple to find the area of the cone parameterized in Example 2. We will define g as a vector expression. Note the syntax when we take the partial derivatives of the vector expression. The command `map` allows Maple to differentiate on each component separately. It is best to wait until after the length of the fundamental cross product has been computed before trying to simplify. Then you are dealing with an expression rather than a vector expression, which is less likely to cooperate.

```
> restart:  with(student):  with(linalg):
> g:=vector([u*cos(v),u*sin(v),u]);
```

$$g := [u\cos(v), u\sin(v), u]$$

```
> gu:=map(diff,g,u);
```

$$gu := [cos(v), sin(v), 1]$$

```
> gv:=map(diff,g,v);
```

$$gv := [-u\sin(v), u\cos(v), 0]$$

```
> fcp:=crossprod(gu,gv);
```

$$fcp := [-4\cos(v), -u\sin(v), \cos(v)^2 u + \sin(v)^2 u]$$

```
> grand:=norm(fcp,2);
```

$$grand := \sqrt{|u\cos(v)|^2 + |u\sin(v)|^2 + |\cos(v)^2 u + \sin(v)^2 u|^2}$$

```
> grand:=simplify(grand,symbolic);
```

$$grand := \sqrt{2}\, u$$

```
> surfacearea:=Doubleint(grand,u=0..2,v=0..2*Pi);
```

$$surfacearea := \int_0^{2\pi} \int_0^2 \sqrt{2}\, u\, du\, dv$$

```
> surfacearea=value(surfacearea);
```

$$surfacearea := 4\pi\sqrt{2}$$

Problems: Do these problems by pencil and paper.

1. Compute the surface area of the curved surface of the cylinder in Example 2 using the approach shown in these notes.

2. Find the area of the portion of the plane that contains the points $P(1,3,1)$, $Q(0,2,3)$, and $Q(1,1,3)$ and is above the triangle with vertices $V_1(1,0,0)$, $V_2(2,1,0)$, $V_3(1,2,0)$.

3. Compute the surface area of the paraboloid $z = 9 - x^2 - y^2, z \geq 0$. Hint: parameterize this surface using polar coordinates. Then $z = 9 - r^2$, $x = r\cos(\theta)$, $y = r\sin(\theta)$ and $D = \{(r,\theta) : \ 0 \leq r \leq 3, 0 \leq \theta \leq 2\pi\}$.

C3M10 Problems: Turn these in as **C3M10**

4. Use Maple to compute the surface area of the paraboloid $z = 9 - x^2 - y^2, z \geq 0$. Hint: parameterize this surface using polar coordinates. Then $z = 9 - r^2$, $x = r\cos(\theta)$, $y = r\sin(\theta)$ and $D = \{(r,\theta) : \ 0 \leq r \leq 3, 0 \leq \theta \leq 2\pi\}$.

5. Use Maple to compute the area of the surface defined by $z = x^2 + y^2$ with domain $D = \{(x,y) : \ 1 \leq x^2 + y^2 \leq 4\}$.

6. Use Maple to compute the area of the hemisphere $z = \sqrt{9 - x^2 - y^2}$ that is inside the vertical cylinder $x^2 + y^2 \leq 4$.